Gravitational catalysis of chiral and color symmetry breaking of quark matter in hyperbolic space

D. Ebert[1,2*], A. V. Tyukov[3], and V. Ch. Zhukovsky[3†]
[1] *Joint Institute for Nuclear Research, Dubna, R-141980, Russia*
[2] *Institut für Physik, Humboldt-Universität zu Berlin, 12489 Berlin, Germany and*
[3] *Faculty of Physics, Department of Theoretical Physics, Moscow State University, 119991, Moscow, Russia*

We study the dynamical breaking of chiral and color symmetries of dense quark matter in the ultrastatic hyperbolic spacetime $R \otimes H^3$ in the framework of an extended Nambu–Jona-Lasinio model. On the basis of analytical expressions for chiral and color condensates as functions of curvature and temperature, the phenomenon of dimensional reduction and gravitational catalysis of symmetry breaking in strong gravitational field is demonstrated in the regime of weak coupling constants. In the case of strong couplings it is shown that curvature leads to small corrections to the flat-space values of condensate and thus enhances the symmetry breaking effects. Finally, using numerical calculations phase transitions under the influence of chemical potential and negative curvature are considered and the phase portrait of the system is constructed.

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I. INTRODUCTION

Dynamical breaking of chiral and color symmetries has been successfully studied within field theories of the Nambu–Jona-Lasinio (NJL) type with four-fermion interactions [1]. They are also quite useful in describing the physics of light mesons (see e.g. [2, 3, 4] and references therein) and diquarks [5, 6].

The possibility for the existence of the color superconductive (CSC) phase with a nonzero colored diquark condensate was proposed both in the region of high baryon densities [7, 8, 9] and moderate densities [10, 11, 12, 13, 14]. In the framework of NJL models the CSC phase formation has generally been considered as a dynamical competition between diquark $\langle qq\rangle$ and usual quark-antiquark condensation $\langle \bar{q}q\rangle$. Special attention has been paid to the catalyzing influence of external fields on chiral symmetry breaking (χSB) in the regime of weak coupling [15, 16, 17] (constant magnetic field), [18, 19] (chromomagnetic fields) and on the condensation of diquarks [20, 21] (chromomagnetic fields). In particular, it was demonstrated that in a strong field the considered symmetry is dynamically broken for an arbitrary weak attraction between quarks. The physical explanation for this is that the effect of dynamical symmetry breaking is accompanied by an effective lowering of dimensionality in strong fields, where the number of reduced units of dimensions depends on the concrete type of the background field. Dynamical symmetry breaking in a magnetic field in spacetimes of dimension higher than four was also considered in [22].

For cosmological and astrophysical applications, it is also interesting to study the influence of spacetime curvature on symmetry breaking. One of the ways to account for the effects of gravity is to use the adiabatic expansion of Green's functions in the vicinity of a fixed point in powers of small curvature (see, for example, the review [23]). However, since second-order phase transitions take place in the infrared region, the considered processes may become sensitive to the global structure of spacetime, and then one needs exact expressions for the propagators in the curved spacetime.

In particular, exact solution can be found for spacetime with high symmetry. A variety of examples of χSB in different symmetric spaces both at zero and finite temperature has been considered in the literature (see e.g. [23]). One of the well-known examples of spaces with constant positive curvature is the Einstein universe of the form $R \otimes S^3$. χSB at finite temperature and chemical potential in the static Einstein universe was recently considered in [24]. Further investigations of quark matter in this gravitational background concerning, in particular, diquark and pion condensation were performed in [25, 26]. There, the positive curvature was shown to lead to the restoration of broken symmetries, thus acting in a similar way as the temperature. Another symmetric space, where exact solution may be found, is the ultrastatic hyperbolic spacetime $R \otimes H^3$. It was noted that in spaces with negative curvature the chiral symmetry is broken even at weak coupling constant (see e.g. [23]). The detailed analysis of the heat kernel in hyperbolic space showed that the physical reason for this phenomenon is the effective dimensional reduction for

* debert@physik.hu-berlin.de
† zhukovsk@phys.msu.ru

fermions in the infrared region [27]. Recently, the combined influence of gravitational and magnetic fields on χSB in the special case of the 2D space of negative curvature, i.e., on the Lobachevsky plane, was considered in [28]. As discussed by these authors, the study of the effects of surface curvature may be important for some condensed matter systems concerning, in particular, the quantum Hall effect in graphene [29, 30].

The aim of this paper is to study the effects of dynamical breaking of chiral and/or color symmetries in dense quark matter under the influence of negative curvature of ultrastatic hyperbolic spacetime $R \otimes H^3$. In the framework of an extended Nambu–Jona-Lasinio model, including $(\bar{q}q)$- and (qq)-interactions, an exact in curvature expression for the thermodynamic potential is derived, which contains all the necessary information about the condensates. Basing upon the analytical solutions of gap equations for quark and diquark condensates we show that in strong gravitational field there arises a gravitational catalysis of dynamical symmetry breaking and chiral and color symmetries may be simultaneously broken even for weakly interacting quarks. This situation resembles the influence of magnetic or chromomagnetic fields on symmetry breaking in flat case. We also consider the role of finite temperature and find that for any fixed value of curvature there exists a critical temperature at which chiral and color symmetries become restored. Moreover, using numerical calculations, phase transitions under the influence of a chemical potential are investigated and the phase portrait of the system at zero temperature is constructed. Finally, it is shown that in the strong coupling regime negative curvature enhances the values of condensates as compared to the flat case. Our analysis demonstrates that negative curvature acts on chiral and color condensates in a way similar to that of a magnetic field in flat spacetime.

II. THE EXTENDED NJL MODEL IN CURVED SPACETIME

Let us briefly remind the basic definitions necessary for the description of fermions in curved spacetime. In 4-dimensional curved spacetime with signature $(+,-,-,-)$, the line element is written as

$$ds^2 = \eta_{\hat{a}\hat{b}} e^{\hat{a}}_{\mu} e^{\hat{b}}_{\nu} dx^{\mu} dx^{\nu}.$$

The gamma-matrices γ_μ, metric $g_{\mu\nu}$ and the vierbein $e^{\mu}_{\hat{a}}$, as well as the definitions of the spinor covariant derivative ∇_ν and spin connection $\omega^{\hat{a}\hat{b}}_{\nu}$ are given by the following relations [31, 32]:

$$\begin{aligned}
&\{\gamma_\mu(x), \gamma_\nu(x)\} = 2g_{\mu\nu}(x), \quad \{\gamma_{\hat{a}}, \gamma_{\hat{b}}\} = 2\eta_{\hat{a}\hat{b}}, \quad \eta_{\hat{a}\hat{b}} = \mathrm{diag}(1,-1,-1,-1),\\
&g_{\mu\nu} g^{\nu\rho} = \delta^{\rho}_{\mu}, \quad g^{\mu\nu}(x) = e^{\mu}_{\hat{a}}(x) e^{\nu\hat{a}}(x), \quad \gamma_\mu(x) = e^{\hat{a}}_{\mu}(x)\gamma_{\hat{a}}. \qquad (1)\\
&\nabla_\mu = \partial_\mu + \Gamma_\mu, \quad \Gamma_\mu = \frac{1}{2}\omega^{\hat{a}\hat{b}}_{\mu}\sigma_{\hat{a}\hat{b}}, \quad \sigma_{\hat{a}\hat{b}} = \frac{1}{4}[\gamma_{\hat{a}}, \gamma_{\hat{b}}],\\
&\omega^{\hat{a}\hat{b}}_{\mu} = \frac{1}{2} e^{\hat{a}\lambda} e^{\hat{b}\rho}[C_{\lambda\rho\mu} - C_{\rho\lambda\mu} - C_{\mu\lambda\rho}], \quad C_{\lambda\rho\mu} = e^{\hat{a}}_{\lambda}\partial_{[\rho}e_{\mu]\hat{a}}. \qquad (2)
\end{aligned}$$

Here, the index $\hat{a}$ refers to the flat tangent space defined by the vierbein at the spacetime point x, and the $\gamma^{\hat{a}}(\hat{a} = 0,1,2,3)$ are the usual Dirac gamma-matrices of Minkowski spacetime. Moreover γ_5, is defined, as usual (see, e.g., [31, 33, 34]), i.e. to be the same as in flat spacetime and thus independent of spacetime variables.

The extended NJL model which includes the $(\bar{q}q)$- and (qq)-interactions of colored up- and down-quarks can be used to describe the formation of the color superconducting phase. For the color group $SU_c(3)$ its Lagrangian takes the form

$$\mathcal{L} = \bar{q}\left[i\gamma^\mu\nabla_\mu + \mu\gamma^0\right]q + \frac{G_1}{2N_c}\left[(\bar{q}q)^2 + \left(\bar{q}i\gamma^5\vec{\tau}q\right)^2\right] + \frac{G_2}{N_c}\left[i\bar{q}_c\varepsilon\epsilon^b\gamma^5 q\right]\left[i\bar{q}\varepsilon\epsilon^b\gamma^5 q_c\right]. \qquad (3)$$

Here, $N_c = 3$ is the number of colors, G_1 and G_2 are coupling constants (their particular values will be chosen in what follows), μ is the quark chemical potential, $q_c = C\bar{q}^t, \bar{q}_c = q^t C$ are charge-conjugated bispinors (t stands for the transposition operation). The charge conjugation operation is defined with the help of the operator $C = i\gamma^{\hat{2}}\gamma^{\hat{0}}$, where the flat-space matrices $\gamma^{\hat{2}}$ and $\gamma^{\hat{0}}$ are used (see, e.g., [31]). The quark field $q \equiv q_{i\alpha}$ is a doublet of flavors and triplet of colors with indices $i = 1,2$; $\alpha = 1,2,3$. Moreover, $\vec{\tau} \equiv (\tau^1, \tau^2, \tau^3)$ denote Pauli matrices in the flavor space; $(\varepsilon)^{ik} \equiv \varepsilon^{ik}$, $(\epsilon^b)^{\alpha\beta} \equiv \epsilon^{\alpha\beta b}$ are the totally antisymmetric tensors in the flavor and color spaces, respectively.

Next, by applying the usual bosonization procedure, we obtain the linearized version of the Lagrangian (3) with collective boson fields σ, $\vec{\pi}$ and Δ,

$$\tilde{\mathcal{L}} = \bar{q}\left[i\gamma^\mu\nabla_\mu + \mu\gamma^0\right]q - \bar{q}\left(\sigma + i\gamma^5\vec{\tau}\vec{\pi}\right)q - \frac{3}{2G_1}(\sigma^2 + \vec{\pi}^2) - \frac{3}{G_2}\Delta^{*b}\Delta^b - \Delta^{*b}\left[iq^t C\varepsilon\epsilon^b\gamma^5 q\right] - \Delta^b\left[i\bar{q}\varepsilon\epsilon^b\gamma^5 C\bar{q}^t\right]. \qquad (4)$$

The Lagrangians (3) and (4) are equivalent, as can be seen by using the Euler-Lagrange equations for the bosonic fields, from which it follows that

$$\Delta^b = -\frac{G_2}{3} i q^t C \varepsilon \epsilon^b \gamma^5 q, \quad \sigma = -\frac{G_1}{3} \bar{q} q, \quad \vec{\pi} = -\frac{G_1}{3} \bar{q} i \gamma^5 \vec{\tau} q. \tag{5}$$

The fields σ and $\vec{\pi}$ are color singlets, and Δ^b is a color anti-triplet and flavor singlet.

In what follows, it is convenient to consider the effective action for boson fields, which is expressed through the integral over quark fields

$$\exp\{\, i S_{\rm eff}(\sigma, \vec{\pi}, \Delta^b, \Delta^{*b})\} = N \int [dq][d\bar{q}] \exp\Big\{\, i \int d^4x \sqrt{-g} \tilde{\mathcal{L}} \Big\}, \tag{6}$$

where

$$S_{\rm eff}(\sigma, \vec{\pi}, \Delta^b, \Delta^{*b}) = -\int d^4x \sqrt{-g} \left[\frac{3(\sigma^2 + \vec{\pi}^2)}{2G_1} + \frac{3\Delta^b \Delta^{*b}}{G_2} \right] + S_q, \tag{7}$$

with S_q standing for the quark contribution to the effective action.

In the mean field approximation, the fields σ, $\vec{\pi}$, Δ^b, Δ^{*b} can be replaced by their ground state averages: $\langle\sigma\rangle$, $\langle\vec{\pi}\rangle$, $\langle\Delta^b\rangle$ and $\langle\Delta^{*b}\rangle$, respectively. Let us choose the following ground state of our model:

$$\langle\Delta^1\rangle \;=\; \langle\Delta^2\rangle \;=\; \langle\vec{\pi}\rangle \;=\; 0.$$

If $\langle\sigma\rangle \;\neq\; 0$, the chiral symmetry is broken dynamically, and if $\langle\Delta^3\rangle \;\neq\; 0$, the color symmetry is broken. Evidently, this choice breaks the color symmetry down to the residual group $SU_c(2)$. (In the following we denote $\langle\sigma\rangle$, $\langle\Delta^3\rangle \;\neq\; 0$ by letters σ, Δ.)

The quark contribution has the following form (for more details see [21])

$$S_q(\sigma, \Delta) \;=\; -i \ln {\rm Det} \left[i\hat{\nabla} - \sigma + \mu\gamma^0 \right] - \tfrac{i}{2} \ln {\rm Det} \left[4|\Delta|^2 + (-i\hat{\nabla} - \sigma + \mu\gamma^0)(i\hat{\nabla} - \sigma + \mu\gamma^0) \right]. \tag{8}$$

Here, the first determinant is over spinor, flavor and coordinate spaces, and the second one is over the two-dimensional color space as well, and $\hat{\nabla} = \gamma^\mu \nabla_\mu$.

Note that the effective potential of the model, the global minimum point of which will determine the quantities σ and Δ, is given by $S_{\rm eff} = -V_{\rm eff} \int d^4x \sqrt{-g}$, where

$$V_{\rm eff} = \frac{3\sigma^2}{2G_1} + \frac{3\Delta\Delta^*}{G_2} + \tilde{V}_{\rm eff}; \quad \tilde{V}_{\rm eff} = -\frac{S_q}{v}, \quad v = \int d^4x \sqrt{-g}. \tag{9}$$

III. THERMODYNAMIC POTENTIAL

In this work we will consider the ultrastatic spacetime $R \otimes H^3$ with constant negative curvature. The metric is given by

$$ds^2 = dt^2 - a^2(d\theta^2 + \sinh^2\theta d\Omega_2), \tag{10}$$

where a is the radius of the hyperboloid which is related to the scalar curvature by the relation $R = -\frac{6}{a^2}$, and $d\Omega_2$ is the metric on the two dimensional unit sphere.

Let us next introduce the one-particle Hamiltonian $\hat{H} = -i\vec{\alpha}\vec{\nabla} + \sigma\gamma^0$, where $\vec{\alpha} = \gamma^0\vec{\gamma}$. Using this operator the quark contribution to the effective action can be written in the following form (for more details see [25]):

$$S_q = -\frac{i}{2} \left\{ \ln {\rm Det} \left[\hat{H}^2 - (\hat{p}_0 + \mu)^2 \right] + 2 \ln {\rm Det} \left[4|\Delta|^2 + (\hat{H} - \mu)^2 - \hat{p}_0^2 \right] \right\}, \tag{11}$$

where $\hat{p}_0 = i\partial_0$, and we have summed over colors (the Det-operator now does not include color space).

In order to calculate the effective action one needs to solve the equation for eigenfunctions of the Hamiltonian $\hat{H}$ in hyperbolic space

$$(-i\vec{\alpha}\vec{\nabla} + \sigma\gamma^0)\Psi_\lambda = \lambda\Psi_\lambda. \tag{12}$$

Decomposing Ψ_λ into upper $\psi_{\lambda,1}$ and lower $\psi_{\lambda,2}$ components and further separating variables by setting $\psi_{1,2} = f_{1,2}(\theta)\chi_{1,2}$, the solutions can be found in terms of hypergeometric functions (for details see e.g. [27, 36] and [34, 35]).

Let us now consider the diagonal element:

$$\mathrm{tr}\langle\vec{x}|\ln\left[\hat{H}^2-(\hat{p}_0+\mu)^2\right]|\vec{x}\rangle = \int_0^\infty d\rho\sum_{\xi=\pm}\sum_{slm}\ln\left[(\xi\lambda(\rho))^2-(\hat{p}_0+\mu)^2\right]\psi^{(s)}_{\xi\rho lm}(\vec{x})^\dagger\psi^{(s)}_{\xi\rho lm}(\vec{x}), \tag{13}$$

where l, m, $\rho = a\sqrt{\lambda^2-\sigma^2}$ and s are the quantum numbers which characterize the spinor χ. Here ξ is the sign of the energy and we will now take $\lambda(\rho) > 0$. Notice that the last sum does not depend on the point $\vec{x}$ due to the homogeneity of the space H^N. Therefore it can be evaluated at the origin $\theta = 0$, when only the $l = 0$ term survives [34] (for generality, we quote below the formulae referring to the N-dimensional space):

$$\sum_{slm}\psi^{(s)}_{\xi\rho lm}(0)^\dagger\psi^{(s)}_{\xi\rho lm}(0) = 2^{\frac{N-1}{2}}\mu_N(\rho), \quad \mu_N(\rho)\equiv\frac{1}{a^N}\frac{\Gamma(N/2)2^{N-3}}{\pi^{N/2+1}}|C_{l=0}(\rho)|^{-2}, \tag{14}$$

where

$$C_l(\rho) = \frac{2^{N-2}}{\sqrt{\pi}}\frac{\Gamma(l+N/2)\Gamma(i\rho+1/2)}{\Gamma(i\rho+l+N/2)}. \tag{15}$$

Notice that the definition of the measure $\mu_N(\rho)$ (the density of states) coincides with that used in [36].

Thus we can calculate the contribution from the first term in (11):

$$\begin{aligned}\ln\mathrm{Det}\left[\hat{H}^2-(\hat{p}_0+\mu)^2\right] &= \mathrm{Tr}\ln\left[\hat{H}^2-(\hat{p}_0+\mu)^2\right] = N_f\int d^Nx\,dt\ \mathrm{tr}\langle\vec{x},t|\ln[\hat{H}^2-(\hat{p}_0+\mu)^2]|\vec{x},t\rangle \\ &= vN_f2^{\frac{N-1}{2}}\int\frac{dp_0}{2\pi}\sum_{\xi=\pm}\int_0^\infty d\rho\,\mu_N(\rho)\ln\left[(\xi\lambda(\rho))^2-(p_0+\mu)^2\right],\end{aligned} \tag{16}$$

where v is the spacetime volume of $R\otimes H^N$, and $N_f = 2$ is the number of flavours.

The second term gives:

$$\ln\mathrm{Det}\left[4|\Delta|^2+(\hat{H}-\mu)^2-\hat{p}_0^2\right] = vN_f2^{\frac{N-1}{2}}\int\frac{dp_0}{2\pi}\sum_{\xi=\pm}\int_0^\infty d\rho\,\mu_N(\rho)\ln\left[4|\Delta|^2+(\lambda(\rho)+\xi\mu)^2-p_0^2\right]. \tag{17}$$

Thus the quark contribution to the effective potential reads

$$\tilde{V}_{\text{eff}} = -\frac{S_q}{v} = \frac{i}{2}N_f2^{\frac{N-1}{2}}\int\frac{dp_0}{2\pi}\sum_{\xi=\pm}\int_0^\infty d\rho\,\mu_N(\rho)\Big\{\ln\left[(\lambda(\rho)+\xi\mu)^2-p_0^2\right]+2\ln\left[4|\Delta|^2+(\lambda(\rho)+\xi\mu)^2-p_0^2\right]\Big\}. \tag{18}$$

In the case of finite temperature $T = 1/\beta > 0$, the following substitutions should be made:

$$\int\frac{dp_0}{2\pi}(\cdots)\to\frac{i}{\beta}\sum_n(\cdots),\quad p_0\to i\omega_n,\quad \omega_n=\frac{2\pi}{\beta}\left(n+\frac{1}{2}\right),\quad n=0,\pm1\pm2,\ldots,$$

where ω_n is the Matsubara frequency. Then the quark contribution to the effective potential (18) becomes the thermodynamic potential Ω_q:

$$\Omega_q = -\frac{2^{\frac{N-1}{2}}N_f}{2\beta}\sum_{n=-\infty}^{+\infty}\sum_{\xi=\pm}\int_0^\infty d\rho\,\mu_N(\rho)\Big\{\ln\left[(\lambda(\rho)+\xi\mu)^2+\omega_n^2\right]+2\ln\left[4|\Delta|^2+(\lambda(\rho)+\xi\mu)^2+\omega_n^2\right]\Big\}. \tag{19}$$

Summing over the Matsubara frequencies we obtain the thermodynamic potential:

$$\begin{aligned}\Omega(\sigma,\Delta) &= N_c\left(\frac{\sigma^2}{2G_1}+\frac{|\Delta|^2}{G_2}\right)-2^{\frac{N-1}{2}}N_f(N_c-2)\int_0^\infty d\rho\,\mu_N(\rho)\left\{\lambda(\rho)+T\ln\left(1+e^{-\beta(\lambda(\rho)-\mu)}\right)+T\ln\left(1+e^{-\beta(\lambda(\rho)+\mu)}\right)\right\} \\ &\quad -2^{\frac{N-1}{2}}N_f\int_0^\infty d\rho\,\mu_N(\rho)\Big\{\sqrt{(\lambda(\rho)-\mu)^2+4|\Delta|^2}+\sqrt{(\lambda(\rho)+\mu)^2+4|\Delta|^2}+ \\ &\quad +2T\ln\left(1+e^{-\beta\sqrt{(\lambda(\rho)-\mu)^2+4|\Delta|^2}}\right)+2T\ln\left(1+e^{-\beta\sqrt{(\lambda(\rho)+\mu)^2+4|\Delta|^2}}\right)\Big\}.\end{aligned} \tag{20}$$

The spectrum of the Dirac operator λ depends only on one dimensionless parameter ρ. Instead of ρ one can introduce the quantity with the dimension of momentum $p = \rho/a$. Then the spectrum of the Dirac operator may be written as

$$\lambda(p) = \sqrt{p^2 + \sigma^2} \equiv E_p. \tag{21}$$

In the case of the 3D space, $N = 3$, the density of states is

$$\mu_3(\rho) = \frac{\rho^2 + \frac{1}{4}}{2\pi^2 a^3}. \tag{22}$$

Thus the thermodynamic potential in the $R \otimes H^3$ spacetime becomes[1]:

$$\begin{aligned} \Omega(\sigma, \Delta) = 3\left(\frac{\sigma^2}{2G_1} + \frac{|\Delta|^2}{G_2}\right) \; - \; & \frac{2}{\pi^2}\int_0^\infty dp \; (p^2 + \frac{1}{4a^2})\big\{E_p + T\ln\left(1 + e^{-\beta(E_p-\mu)}\right) + T\ln\left(1 + e^{-\beta(E_p+\mu)}\right) \\ & + \; \sqrt{(E_p-\mu)^2 + 4|\Delta|^2} + \sqrt{(E_p+\mu)^2 + 4|\Delta|^2} \\ & + \; 2T\ln\left(1 + e^{-\beta\sqrt{(E_p-\mu)^2+4|\Delta|^2}}\right) + 2T\ln\left(1 + e^{-\beta\sqrt{(E_p+\mu)^2+4|\Delta|^2}}\right)\big\}. \end{aligned} \tag{23}$$

It should be noted that the thermodynamic potential is divergent at large p. Therefore we must use some regularization procedure. Since we interpret p as the module of the momentum, the easiest way to regularize the divergent integral is to introduce the momentum cutoff, $p \le \Lambda$. Then the regularized potential looks as follows:

$$\begin{aligned} \Omega^{\text{reg}}(\sigma, \Delta) = 3\left(\frac{\sigma^2}{2G_1} + \frac{|\Delta|^2}{G_2}\right) \; - \; & \frac{2}{\pi^2}\int_0^\Lambda dp \; (p^2 + \frac{|R|}{24})\big\{E_p + T\ln\left(1 + e^{-\beta(E_p-\mu)}\right) + T\ln\left(1 + e^{-\beta(E_p+\mu)}\right) \\ & + \; \sqrt{(E_p-\mu)^2 + 4|\Delta|^2} + \sqrt{(E_p+\mu)^2 + 4|\Delta|^2} \\ & + \; 2T\ln\left(1 + e^{-\beta\sqrt{(E_p-\mu)^2+4|\Delta|^2}}\right) + 2T\ln\left(1 + e^{-\beta\sqrt{(E_p+\mu)^2+4|\Delta|^2}}\right)\big\}, \end{aligned} \tag{24}$$

where we used the expression for the scalar curvature $R = -\frac{6}{a^2}$. This formula consists of two parts arising from the two terms in the first round bracket of the integrand: the first part is the same as in (3+1)D Minkowski spacetime, while the second one, which is linear in curvature, corresponds to the contribution from (1+1)D spacetime. Hence, when the contribution of the second term dominates over the first one, dimensional reduction by two units takes place.

The values of condensates σ and $|\Delta|$ correspond to the point of global minimum of the regularized thermodynamic potential and are determined as solutions of the gap equations

$$\frac{\partial\Omega^{\text{reg}}}{\partial\sigma} = 0, \qquad \frac{\partial\Omega^{\text{reg}}}{\partial|\Delta|} = 0. \tag{25}$$

In the following sections we will consider the behavior of the condensates as functions of curvature, temperature and chemical potential.

IV. ANALYTICAL SOLUTIONS

A. Chiral condensate

Let us first consider the case when chiral symmetry is broken while the color symmetry remains unbroken ($\sigma \neq 0$ and $\Delta = 0$). Since the quark condensate appears even in the vacuum, we put for simplicity $\mu = 0$ and $T = 0$.

Then one can obtain the following expression for the effective potential

$$V_{\text{eff}}(\sigma) = \frac{\Lambda^4}{\pi^2}v_0(x), \quad v_0(x) = \frac{3x^2}{2g} - \frac{3}{4}F(x) - \frac{|r|}{8}G(x), \quad x = \frac{\sigma}{\Lambda}, \; g = \frac{\Lambda^2}{\pi^2}G_1, \; r = \frac{R}{\Lambda^2}, \tag{26}$$

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[1] Note that after introducing the quantity with dimension of momentum $p = \rho/a$ the spectrum of the Dirac operator (21) formally coincides with the usual dispersion relation in flat Minkowski spacetime and does not depend on curvature. At the same time, the density of states (22) in the hyperbolic space differs from the usual measure of integration over momentum in Minkowski spacetime by an additional term depending on the curvature (in fact, this is the only curvature dependent part of the thermodynamic potential). To obtain the correct measure of integration over the continuum spectrum one needs to use the properly normalized eigenfunctions in (14).

where

$$F(x) = (2 + x^2)\sqrt{1 + x^2} - x^4 \ln \frac{1 + \sqrt{1 + x^2}}{x}, \quad G(x) = \sqrt{1 + x^2} + x^2 \ln \frac{1 + \sqrt{1 + x^2}}{x}. \tag{27}$$

The gap equation for the condensate σ reads

$$\frac{1}{g} = \sqrt{1 + x^2} + (\frac{|r|}{12} - x^2) \ln \frac{1 + \sqrt{1 + x^2}}{x}. \tag{28}$$

An analytical solution of this equation can be obtained only for small $x \ll 1$ ($\sigma \ll \Lambda$). Expanding the right hand side of (28) in x we have

$$\left(\frac{1}{g} - 1\right) = (\frac{|r|}{12} - x^2) \ln \frac{2}{x}. \tag{29}$$

We will consider three different cases: a) subcritical g, $g < g_c = 1$, where g_c is the critical constant in flat four dimensional spacetime; b) near critical g, when $g \to g_c - 0$ and c) overcritical g, $g > 1$.

In the case of subcritical g, a nontrivial solution of the gap equation (29) exists only if $x^2 < \frac{|r|}{12}$. In the strong curvature limit $\sigma^2 \ll \frac{|R|}{12}$, the chiral condensate is given by

$$\sigma_0 = 2\Lambda \exp\left[-\frac{12}{|r|}\left(\frac{1}{g} - 1\right)\right] = 2\Lambda \exp\left[-\frac{12\pi^2(1 - g)}{|R|G_1}\right]. \tag{30}$$

One can distinguish two sub-cases in which the last expression is consistent with the above made assumptions:
i) $g < 1$, $r \ll 1$ or ii) $g \ll 1$, $r \sim 1$. The case ii) shows that the gap equation has a nontrivial solution even at arbitrary weak coupling constant.

The expression (30) looks very similar to the chiral condensate in the two-dimensional Gross-Neveu model. After excluding the coupling constant from the effective potential by using the gap equation (29), we obtain

$$V_{\text{eff}}(\sigma) = \frac{|R|\sigma^2}{16\pi^2}\left(\ln \frac{\sigma^2}{\sigma_0^2} - 1\right), \tag{31}$$

and this is indeed (up to a dimensional factor) the effective potential of the Gross-Neveu model. Hence, we conclude that in the case of subcritical coupling the strong gravitational field of hyperbolic space leads to the effective dimensional reduction from (3+1) to (1+1) (see also [27]).

One should also note that the non-analytical dependence of the chiral condensate on curvature in the exponent of (30) looks quite similar to that in a magnetic field

$$\sigma_0 = \sqrt{\frac{|eB|}{\pi}} \exp\left[-\frac{2\pi^2(1 - g)}{|eB|G_1}\right], \tag{32}$$

but the pre-exponential factors differ (see for example the review [37]). This fact demonstrates that in the case of hyperbolic space the negative curvature plays an analogous catalyzing role as the magnetic field does in flat space, where it gives rise to the catalysis of χSB even at arbitrary weak attraction between quarks.

In the near critical regime $g \to g_c - 0$ the chiral condensate is just the constant

$$\sigma_0 = \sqrt{\frac{|R|}{12}}, \tag{33}$$

where the curvature must be small $\frac{|R|}{12} \ll \Lambda^2$.

In the overcritical regime $g > 1$, a nontrivial solution of (29) exists only if $x^2 > \frac{|r|}{12}$. In the weak curvature limit $\frac{|R|}{12} \ll m^2$ we obtain

$$\sigma_0 = m\left(1 + \frac{|R|}{24m^2} + O\left(\frac{R^2}{m^4}\right)\right), \tag{34}$$

where m is the solution of the gap equation at $R = 0$. It is seen that in this case the curvature leads to small analytical corrections to the flat-space value of chiral condensate.

Next, let us consider the influence of finite temperature on the behavior of the chiral condensate. The temperature dependent contribution to the thermodynamic potential looks as follows

$$\Omega_T(\sigma) = -\frac{12}{\pi^2}T\int_0^\infty dp\left(p^2+\frac{|R|}{24}\right)\ln\left(1+e^{-\beta E_p}\right).$$

Expanding the logarithm into a series and performing the integration over momentum, we obtain

$$\Omega_T(\sigma) = \frac{12}{\pi^2}T\sigma\left(\frac{|R|}{24}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}K_1(n\beta\sigma)+\sigma^2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{K_2(n\beta\sigma)}{n\beta\sigma}\right),$$

where $K_\nu(x)$ is the Macdonald function (modified Bessel function). Then the gap equation at finite temperature reads

$$\frac{1}{g} = \sqrt{1+x^2}+(\frac{|r|}{12}-x^2)\ln\frac{1+\sqrt{1+x^2}}{x}-\frac{|r|}{12}I_1-x^2(I_3-I_1), \tag{35}$$

where

$$I_1(\beta\sigma) = -2\sum_{n=1}^{\infty}(-1)^nK_0(n\beta\sigma), \quad I_3(\beta\sigma) = -2\sum_{n=1}^{\infty}(-1)^nK_2(n\beta\sigma).$$

Here we consider only the most interesting case of subcritical coupling and strong gravitational field. Thus, as previously at $g<1$ and $\sigma^2 \ll \frac{|R|}{12}$, we obtain for the chiral condensate

$$\sigma_0(T) = 2\Lambda\exp\left[-\frac{12\pi^2(1-g)}{|R|G_1}-I_1(\beta\sigma_0(T))\right] = \sigma_0(0)\exp\left[-I_1(\beta\sigma_0(T))\right]. \tag{36}$$

The function $I_1(x)$ is positive and monotonically decreases which means that temperature leads to the restoration of broken symmetry. The critical temperature T_c is defined by the condition $\sigma_0(T_c)=0$ which gives the BCS-like relation

$$T_c = \pi^{-1}e^C\sigma_0(0) \simeq 0,57\sigma_0(0), \tag{37}$$

where $\sigma_0(0)$ is given by (30). It should be mentioned that the temperature dependence of the chiral condensate (36) is the same as in a constant chromomagnetic field [20].

This brief analysis shows that in the case of subcritical coupling the negative curvature leads to the catalysis of χSB while in the overcritical regime it leads to the enhancement of the chiral condensate. One might expect the same behaviour in the case of a color condensate. The case of overcritical coupling will also be considered in the next section using numerical methods.

B. Mixed phase

Let us now consider a more general situation, when both condensates can take nonzero values. What concerns color symmetry breaking in flat spacetime, it is well known that diquark pairing and color superconductivity arise (for large quark number densities and, correspondingly, at large chemical potential) due to an instability of the Fermi surface so that the color superconducting state is energetically more preferable. For studying the influence of gravity on the formation of a color condensate in its most "pure form", we find it convenient, contrary to the flat case, in the following to take $\mu=0$. Then the effective potential at zero temperature can be written in the form

$$V_{\rm eff}(\sigma,\Delta) = \frac{\Lambda^4}{\pi^2}v_0(x,y), \quad v_0(x) = \frac{3A}{2}x^2+By^2-\frac{1}{4}\left(F(x)+2F(z)\right)-\frac{|r|}{24}\left(G(x)+2G(z)\right), \tag{38}$$

where

$$x=\frac{\sigma}{\Lambda}, \quad y=\frac{2|\Delta|}{\Lambda}, \quad z=\frac{m_*}{\Lambda}=\sqrt{x^2+y^2}, \quad A=\frac{1}{g}=\frac{\pi^2}{\Lambda^2G_1}, \quad B=\frac{3\pi^2}{4\Lambda^2G_2}, \quad r=\frac{R}{\Lambda^2}, \tag{39}$$

and the functions F and G are the same as in (27). The nontrivial solutions for condensates satisfy the following gap equations obtained from (38)

$$3A = H(x) + 2H(z) + \frac{|r|}{12}(K(x) + 2K(z)), \tag{40}$$

$$B = H(z) + \frac{|r|}{12}K(z), \tag{41}$$

where

$$H(x) = \sqrt{1+x^2} - x^2 \ln \frac{1+\sqrt{1+x^2}}{x}, \quad K(x) = \ln \frac{1+\sqrt{1+x^2}}{x}. \tag{42}$$

Substituting (41) in (40), we obtain a separate equation for the chiral condensate

$$3A - 2B = H(x) + \frac{|r|}{12}K(x), \tag{43}$$

which formally coincides with (28) in the previous Subsection **A**, when $1/g$ becomes replaced by $(3A-2B)$. As in the previous case, we will solve equations (43) and (41) in the limit of small condensates $x \ll 1$ and $y \ll 1$ and consider only the most interesting case of strong curvature $x^2 \ll \frac{|r|}{12}$, $y^2 \ll \frac{|r|}{12}$ and subcritical couplings $A > 1$, $B > 1$, where the effect of gravitational catalysis is more clearly seen. In this limit we simply have $H(x) = 1$ and $K(x) = \ln(2/x)$. The solutions of the gap equations are as follows

$$\sigma_0^2 = 4\Lambda^2 \exp\left[-\frac{24}{|r|}(3A - 2B - 1)\right], \tag{44}$$

$$m_{*0}^2 = \sigma_0^2 + 4|\Delta_0|^2 = 4\Lambda^2 \exp\left[-\frac{24}{|r|}(B-1)\right], \tag{45}$$

where the coupling constants must satisfy the inequalities: $3A - 2B > 1$ and $B > 1$. From this, the color condensate follows

$$4|\Delta_0|^2 = m_{*0}^2 \left\{1 - \exp\left[-\frac{24}{|r|}(3A - 3B)\right]\right\}, \tag{46}$$

which exists only if $A > B$ (i.e. $G_2 > \frac{3}{4}G_1$). Hence, both condensates may exist simultaneously in the region $A > B > 1$ (in this region the inequality $3A - 2B = A + 2(A-B) > 1$ holds automatically).

Excluding the coupling constants from the effective potential one obtains

$$V_{\text{eff}}(\sigma, \Delta) = \frac{|R|}{48\pi^2}\left[\sigma^2\left(\ln\frac{\sigma^2}{\sigma_0^2} - 1\right) + 2m_*^2\left(\ln\frac{m_*^2}{m_{*0}^2} - 1\right)\right] \tag{47}$$

The other possible stationary point of the effective potential is $\sigma = 0$ and $\Delta \neq 0$. This type of solution may be obtained from the previous one by letting $\sigma_0^2 = 0$ in (45) so that the combined condensate m_{*0}^2 reduces to the tilded expression $4|\widetilde{\Delta}_0|^2$.

The phase structure of the model is defined by the global minimum of the effective potential. There are four types of stationary points of the effective potential: $(0,0)$, $(\widetilde{\sigma}_0, 0)$, $(0, \widetilde{\Delta}_0)$ and (σ_0, Δ_0). Let us denote the corresponding values of the effective potential at these points by v_1, v_2, v_3 and v_4. The effective potential is normalized in such a way that $v_1 = 0$. The other values are

$$v_4 = -\frac{|R|}{48}\left[\sigma_0^2 + 2m_{*0}^2\right], \quad v_3 = -\frac{|R|}{6}|\widetilde{\Delta}_0|^2, \quad v_2 = -\frac{|R|}{16}\widetilde{\sigma}_0^2, \tag{48}$$

where $\widetilde{\sigma}_0$ is given by (30). First of all we see that the minimum $v_1 = 0$ of the symmetry case is higher than the other minima. This means in contrast to the flat case that for subcritical couplings the symmetric phase in hyperbolic space is unstable under the formation of different condensates and symmetry breaking. Since $m_{*0}^2 = 4|\widetilde{\Delta}_0|^2$, it is easily seen that $v_4 < v_3$ in the region $A > B > 1$. The mixed phase is the true vacuum of our model if $v_4 < v_2$ or $\sigma_0^2 + 2m_{*0}^2 > 3\widetilde{\sigma}_0^2$. Dividing both sides of the last inequality by $\widetilde{\sigma}_0^2$ and using (30), (44) and (45) we obtain the following condition

$$\exp\left[\frac{24}{|r|}2(B-A)\right] + 2\exp\left[\frac{24}{|r|}(A-B)\right] > 3.$$

Introducing a new variable $u = \exp\left[\frac{24}{|r|}(B-A)\right]$, $0 < u < 1$ for $A > B > 1$, we see that the above condition is just the simple cubic inequality $u^3 + 2 > 3u$, which is automatically satisfied for $0 < u < 1$. Thus, the condition $A > B > 1$ is sufficient for the mixed phase to be the true vacuum state of our model.

In the opposite case $B \geq A > 1$ the chiral and color condensates cannot exist simultaneously and we need to compare v_2 and v_3. The inequality $v_3 < v_2$ leads to $A - B > \frac{|r|}{24} \ln \frac{3}{2}$ which contradicts our previous assumption. Therefore in the region $B \geq A > 1$ only the phase with broken chiral symmetry occurs, and the chiral condensate is described by (30).

Let us also briefly consider the influence of finite temperature on the mixed phase. Proceeding in the same manner as in Subsection **A**, we obtain the following temperature dependence of condensates

$$\sigma_0^2(T) = \sigma_0^2(0) \exp\left[-2I_1(\beta\sigma_0(T))\right], \tag{49}$$

$$m_{*0}^2(T) = m_{*0}^2(0) \exp\left[-2I_1(\beta m_{*0}(T))\right], \tag{50}$$

where $\sigma_0^2(0)$ and $m_{*0}^2(0)$ are given by (44) and (45), respectively. In analogy with (37), the critical temperatures for chiral and color condensates are

$$T_c^\sigma = \pi^{-1} e^C \sigma_0(0), \qquad T_c^\Delta = \pi^{-1} e^C m_{*0}(0). \tag{51}$$

Since the inclusion of temperature results in a smooth diminution of condensates, the relation between couplings which determines the phase structure of the model should not change. Again both condensates may exist simultaneously if $A > B$ which implies the relation $T_c^\Delta > T_c^\sigma$. Therefore at $T < T_c^\sigma$ the mixed phase is realized with chiral and color condensate of the form

$$|\Delta_0(T)|^2 = \frac{1}{4}\left\{m_{*0}^2(T) - \sigma_0^2(T)\right\}. \tag{52}$$

Obviously, at $T_c^\sigma < T < T_c^\Delta$, chiral symmetry is restored, while the color symmetry remains broken

$$|\Delta_0(T)|^2 = |\widetilde{\Delta}_0(T)|^2 = \frac{m_{*0}^2(T)}{4}. \tag{53}$$

Finally, at $T > T_c^\Delta$ both symmetries become restored.

The phase portraits of the system are presented in Fig.1 for fixed values of coupling constants A and B, for $A \leq B$ (left picture) and for $A > B$ (right picture). The critical curves which separate different phases are described by corresponding formulas (37), (51) for critical temperatures.

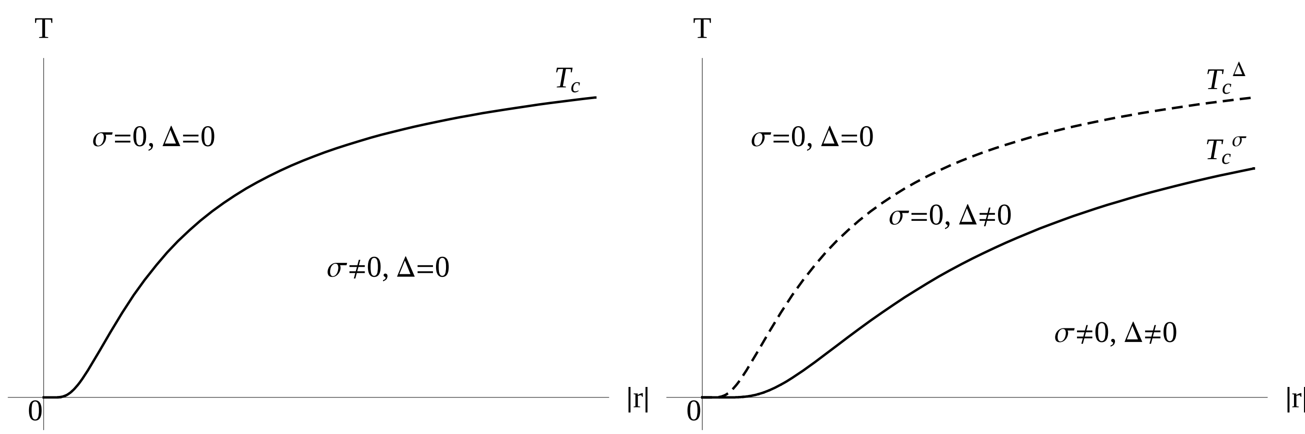

FIG. 1: The phase portraits of the system showing critical curves as functions of $|r|$ for fixed values of couplings A and B: for $A \leq B$ (left) and for $A > B$ (right).

In fact the solutions (49), (50) of the gap equations give only an implicit dependence of the condensates $\sigma_0(T)$ and $m_{*0}(T)$ on temperature. Due to the complexity of the function $I_1(x)$, no analytical solutions for condensates can be found. However, equations (49), (50) can be easily solved by using numerical methods, for example, an iterative procedure. When the temperature is fixed, the result of such procedure will depend only on the first step of iteration. The most obvious choice is to take the condensates at zero temperature as a first approximation. The schematic

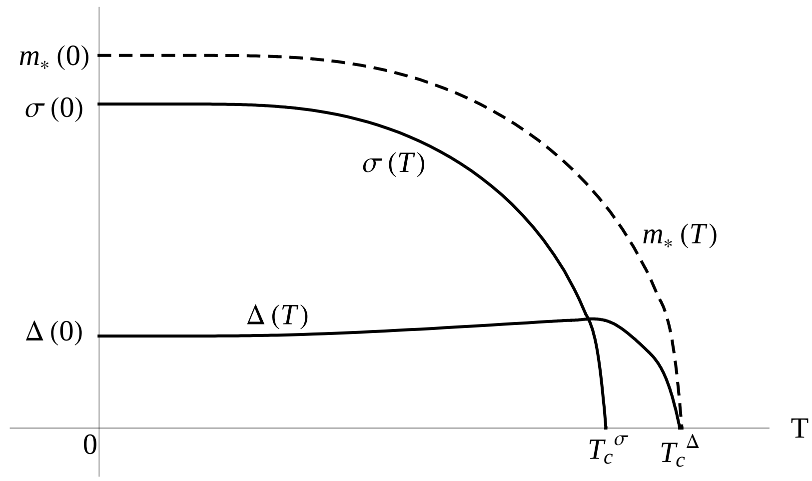

FIG. 2: Condensates σ_0, Δ_0 and m_{*0} as functions of temperature inside the mixed phase. (Henceforth we will omit the subscript 0 of condensates on the plots.)

picture of condensates inside the mixed phase is shown in Fig.2 for fixed values of $\sigma_0(0)$ and $\Delta_0(0)$. It is interesting to note that the condensates σ_0 and m_{*0} have the usual form of the BCS theory of superconductivity and decrease with temperature, while the color condensate Δ_0 slightly grows at $T \lesssim T_c^\sigma$ and has a maximum around T_c^σ. We have not specified the values of condensates on these curves since we have no phenomenologically motivated choice of coupling constants and curvature. Fig.2 demonstrates only qualitatively the behaviour of condensates as functions of the temperature at fixed curvature $|r|$ and couplings A and B. Moreover, since the condensates at nonzero temperature depend on the curvature only through the condensates at zero temperature, the increasing curvature will produce the appropriate stretching of curves in Fig.2.

V. PHASE TRANSITIONS

In the previous section the phase structure of the model was extensively studied in the regime of subcritical couplings by formally considering arbitrary ratios of coupling constants. In particular, in order to investigate the influence of curvature on condensates in "pure form", the chemical potential was taken to be zero. Now, in this section, we turn to the more general case of phase transitions at finite chemical potential (quark number density).

In the following we want to compare our results with the well established case of flat spacetime. Since the role of a growing temperature in the restoration of broken symmetries was already demonstrated, our considerations will be restricted here to the case of zero temperature. The investigation of the phase transitions at finite chemical potential beyond the limit of small condensates is difficult to perform analytically, and hence numerical methods will now be used.

In order to be able to make comparison with the flat case, we consider the following "more physical" relation of couplings taken from the instanton-motivated NJL-model [11] (see also [21]):

$$G_2 = \frac{3}{8} G_1. \tag{54}$$

Let us now fix $g = G_1\Lambda^2/\pi^2 = 1.4$ in such a way that the chiral condensate at zero curvature, $R = 0$, in the vacuum is equal to the usual value of the constituent quark mass in flat space ($\sigma_0 = 350$ MeV). Here the cutoff parameter is taken to be $\Lambda = 600$ MeV.

In terms of dimensionless couplings A and B (see (39)), relation (54) corresponds to $B = 2A$, and one might think that it excludes the existence of a mixed phase in accordance with our previous discussion. However, we should stress that the inequality $B \geq A$, which guaranties the absence of a mixed phase, was obtained only for subcritical couplings. Here we have fixed $g = 1.4$ and thus $A = 1/g < 1$, and the overcritical regime is realized. The difference between these cases is that in the subcritical regime the main contribution to the values of condensates is given by the two-dimensional part of the effective potential, while in the overcritical regime, the condensates receive their contribution mainly from the flat four-dimensional part. Therefore our previous arguments are not applicable in this case.

It was observed earlier that in a flat four dimensional spacetime the relation (54) between coupling constants leads to the absence of a mixed phase in a wide range of parameters [11]. In the previous section, we have found that in the overcritical regime finite curvature gives only small corrections to the flat-space value of chiral condensate (see (34)). Using numerical calculations, we will now analogously show that in a wide range of the values of curvature its contribution to condensates is small in comparison with their values in flat case. It is clear that such corrections can not change the phase structure of the model obtained in flat case in [11], and thus can not lead to formation of new phases. Therefore, in what follows, we assume that there is no mixed phase, where both condensates simultaneously take nonzero values.

As is well-known, for increasing chemical potential there arises diquark pairing, whereas the chiral condensate becomes suppressed.

The corresponding behavior of the chiral and color condensates as functions of the chemical potential at $r = 0$ and $|r| = 1$ is shown in Fig.3.

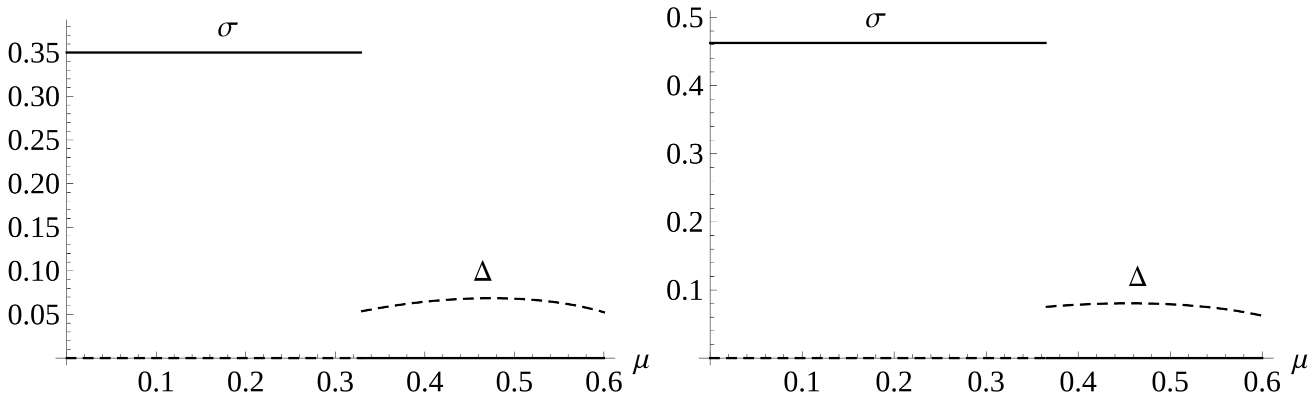

FIG. 3: Condensates σ and Δ as functions of μ (all quantities are given in units of GeV) at $r = 0$ (left) and at $|r| = 1$ (right).

As is seen from Fig.3, the critical chemical potential, at which the phase transition takes place, in flat space, $r = 0$, is $\mu_c \approx 330$ MeV, while at $|r| = 1$ $\mu_c \approx 366$ MeV. From this we can conclude that the critical line $\mu_c(|r|)$ which separates the two phases is a growing function of the curvature. The numerical study of the number density, which is the first derivative of the thermodynamical potential with respect to the chemical potential, shows that it is discontinuous at μ_c. Thus, an increasing chemical potential leads to a first order phase transition.

It should also be mentioned that for the limiting case of zero curvature our results for the value of the critical chemical potential $\mu_c \approx 330$ MeV and the maximum value of the color condensate $\Delta \approx 70$ MeV (see Fig.3) are in agreement with the results obtained in [38] for the same values of σ_0 and Λ (note that our value of the color condensate Δ is by definition two times less than condensate Δ in [38]).

We can also examine the condensates σ and Δ as functions of the absolute value of curvature $|r|$. The behavior of chiral and color condensates inside both phases is presented in Fig.4.

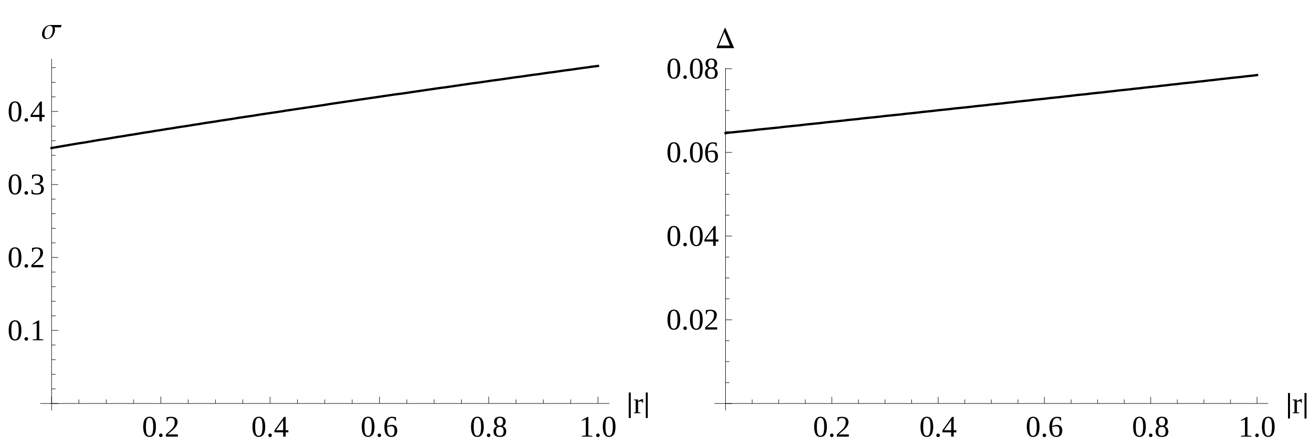

FIG. 4: Condensates (in units of GeV) σ at $\mu = 200$ MeV (left) and Δ at $\mu = 400$ MeV (right) as functions of $|r|$.

It is clear that with growing curvature the values of condensates slowly increase, which results in the enhancement of the symmetry breaking effects. As we have already mentioned above, in a wide range of curvature values the increment of the condensates is small with respect to their values at $r = 0$. Moreover, it is seen from Fig.4 that condensates grow with curvature almost linearly which is in agreement with our previous analytical consideration (see (34)).

Finally, the $r-\mu$ - phase portrait of our system is shown in Fig.5. As it was already mentioned, the critical chemical potential, at which the first order phase transition takes place, slightly grows with increasing curvature.

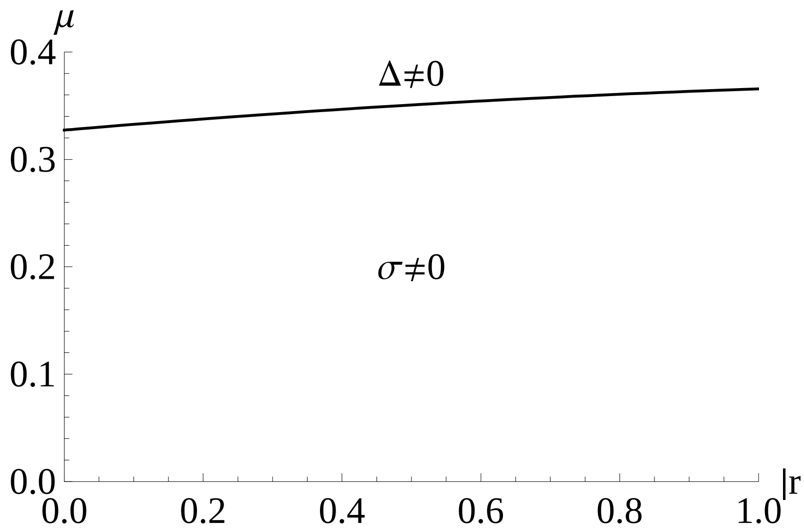

FIG. 5: Phase portrait for the coupling relation $G_2 = \frac{3}{8}G_1$.

VI. SUMMARY

In the framework of an extended NJL model we have investigated chiral and/or color symmetry breaking in dense quark matter under the influence of negative curvature of hyperbolic space by employing the thermodynamic (effective) potential of the system as a function of chiral and diquark condensates, temperature, curvature and chemical potential.

Two different regimes of dynamical symmetry breaking have been studied. First, the regime of subcritical couplings, where the values of coupling constants are lower than their critical values in flat space, was considered. As is well known, in this situation in four dimensions, no symmetry breaking takes place, and gap equations have only trivial solutions. Unlike the flat case, in hyperbolic space, symmetry may, however, be broken even for an arbitrary small coupling constant. In the case of subcritical couplings we obtained expressions for the chiral and color condensates that depend non-analytically on curvature. Secondly, the regime of overcritical coupling constants was investigated, when the symmetry may be broken even in flat space. In this case it was shown that curvature leads to small analytical corrections which increase the flat-space values of condensates and thus enhances the symmetry breaking effects.

It is interesting to note that in the subcritical regime of coupling constants the strong gravitational field of hyperbolic space serves as a catalyzing factor similar to the role of the magnetic field [15, 16, 17] or chromomagnetic fields [18, 19, 20, 21] in the effects of dynamical symmetry breaking. As we have explicitly demonstrated, the gravitational catalysis takes place for chiral and color condensates. It is also worth mentioning that the effect of gravitational catalysis is accompanied by a lowering of dimensions of the system. In the regime of subcritical couplings the solutions of gap equations look quite similar to those for the 2D Gross-Neveu model. Therefore, we concluded that the strong curvature of hyperbolic space leads to an effective dimensional reduction by two units (see also [27]).

As we have already mentioned, in the subcritical regime the negative curvature essentially changes the phase structure of the NJL model found in [39] making the symmetric phase unstable under the formation of condensates, while in the overcritical case one expects only minor modifications to the phase structure obtained in flat space. Therefore we have extensively studied the phase structure in the subcritical regime. It is interesting to note that the overall phase structure depends only on the ratio of the inverse (dimensionless) coupling constants A and B, but not on the curvature. For subcritical couplings the mixed phase is realized only if $A/B > 1$, while if $A/B \leq 1$ only the chiral symmetry may be broken. The same critical ratio $A/B = 1$ was found in the flat space in the framework of the

random matrix model [40]. It was argued by these authors that this relation is a consequence of global symmetries of the model.

Moreover, the influence of finite temperature on phase structure was investigated. The phase portraits of the system for different relations between couplings were constructed. In particular, we have shown that for any fixed value of curvature there exists a critical temperature at which the phase transition takes place and symmetry becomes restored.

Finally, using numerical calculations, we have investigated the phase transitions between χSB and CSC phases under the influence of chemical potential and curvature in the regime of overcritical couplings. It was demonstrated that similar to the flat case there arises a diquark pairing for increasing chemical potential, while the chiral condensate becomes suppressed. The phase portrait of the system at zero temperature was also constructed, and it was shown that the critical line $\mu_c(|r|)$, separating the two different phases, is a growing function of curvature. The chiral and diquark condensates, σ and Δ, acquire only small corrections due to curvature increasing the flat-space values of condensates and this leads to an enhancement of the symmetry breaking effects.

The results of this paper, although describing a model situation with symmetry breaking in a hyperbolic space, may hopefully find further development in more realistic situations with phase transitions in quark matter under the influence of strong gravitational fields.

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